Notes on band-pass spatial filters

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1 Introduction

Different constraints have to be considered:

- sampling and truncation issues
- coverage of radial frequency
- coverage of orientations
- DC sensitivity

2 Gabor function

We consider a vertically oriented Gabor function [1] [2] centered at the origin:

$$f(x, y, \psi) = A \cdot \frac{1}{2\pi\sigma_x\sigma_y} \exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}\right) \operatorname{cis}(k_0 x + \psi), \tag{1}$$

and its Fourier transform:

$$\begin{aligned} F(k_x, k_y, \psi) &= A \cdot \exp\left(-\frac{\sigma_x^2}{2}k_x^2 - \frac{\sigma_y^2}{2}k_y^2\right) \\ &\quad * \exp(j\psi)\delta(k_x - k_0) \end{aligned} \tag{2}$$

where the asterisk $*$ indicates the convolution product, k_0 is the peak tuning frequency, σ_x and σ_y determine the x and y filter dimensions, and ψ is the phase parameter for the sinusoidal modulation, and A is a normalisation constant, which is application dependent. Two alternative normalisation conditions have been used most frequently in the literature:

1. unitary area condition, which corresponds to a unitary maximum value in the frequency domain::

$$\max_{k_x,k_y} F(k_x, k_y, \psi) = 1 \quad \rightarrow \quad A = 1 \tag{3}$$

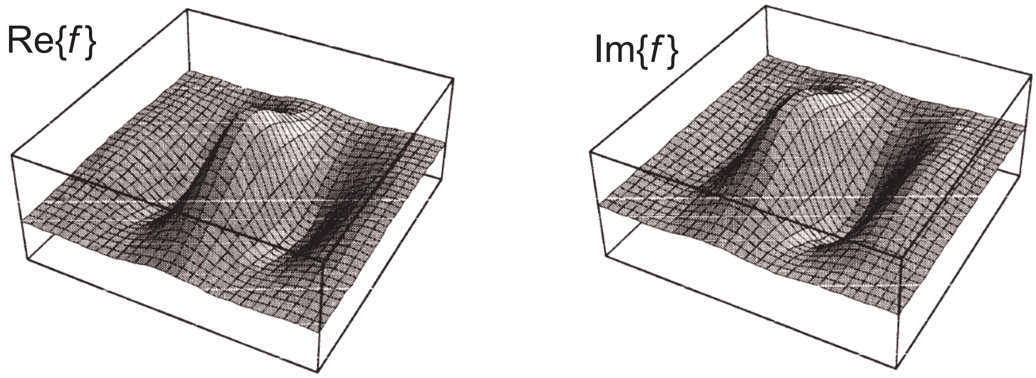

Figure 1: Real and imaginary component of a Gabor function.

2. unitary energy condition:

$$\| f(x,y,\psi) \|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(x,y,\psi) f(x,y,\psi) \mathrm{d}x\mathrm{d}y = 1 \quad \rightarrow \quad A = 2\sqrt{\pi\sigma_x\sigma_y}. \tag{4}$$

We can further characterize the Gabor filter in the frequency domain by:

Absolute bandwidth @ cut-off frequencies (k_x^l, k_x^h) corresponding to half of the amplitude spectrum $F(k_x, k_y, \psi)$:

$$\Delta k = \frac{2\sqrt{2\ln 2}}{\sigma_x} \tag{5}$$

@ cut-off frequencies (k_x^l, k_x^h) at one standard deviation of the amplitude spectrum ($\sigma_f = 1/\sigma_x$):

$$\Delta k = 2\sigma_f \tag{6}$$

Relative bandwidth (in octave)

$$\beta = \log_2\left(\frac{k_x^h}{k_x^l}\right) = \log_2\left(\frac{k_0 + \Delta k/2}{k_0 - \Delta k/2}\right) \tag{7}$$

Typically β is chosen around 1 ($\beta \in [0.8, 1.2]$)[1].

@ cut-off frequencies (k_x^l, k_x^h) corresponding to half of the amplitude spectrum $F(k_x, k_y, \psi)$:

$$\beta = \log_2\left(\frac{\sigma_x k_0 + \sqrt{2\ln 2}}{\sigma_x k_0 - \sqrt{2\ln 2}}\right) \tag{8}$$

[1] $\beta = 1$ allows a good coverage of the frequency space when one adopts a pyramidal multiscale approach (see Section 4).

@ cut-off frequencies (k_x^l, k_x^h) at one standard deviation of the amplitude spectrum ($\sigma_f = 1/\sigma_x$):

$$\beta = \log_2 \frac{k_0 + \sigma_f}{k_0 - \sigma_f} = \log_2 \frac{\sigma_x k_0 + 1}{\sigma_x k_0 - 1} \tag{9}$$

From Eqs.(8) and (9) it is straighforward to show that the spatial "effective" support of the filter with respect to the x axis is:

$$\sigma_x = \frac{1}{k_0}\left(\frac{2^\beta + 1}{2^\beta - 1}\right)\sqrt{2\ln 2} \tag{10}$$

and

$$\sigma_x = \frac{1}{k_0}\left(\frac{2^\beta + 1}{2^\beta - 1}\right), \tag{11}$$

respectively.

3 Sampling and spatial scale

To implement a multiscale approach in an efficient way, a pyramidal approach is suggested [3]. The approach guarantees a reduced computational load and the possibility of making more correct comparisons across the different scales (the filters at the different scales are indeed defined on the same number of samples). Once fixed the filter's parameters for the highest resolution (taking care to meet Nyquist requirements), the outputs at the lower resolutions (scales) can be derived straighforwardly by the pyramid.

By using pixels as units, the sampling period is 1 pixel, that corresponds to $1/2\pi$ pixel^{-1} sampling frequency. Thus, the maximum bandwidth Ω of the signal to avoid aliasing is π pixel^{-1} ($\Omega \leq \pi$ pixel^{-1}). Accordingly, considering the symmetry character of the Gabor spectrum, the maximum peak frequency respect to the Nyquist sampling condition can be derived from the following equation:

$$k_0 + \frac{\Delta k}{2} \leq \pi \tag{12}$$

4 Multiscale frequency space coverage

The pyramidal approach corresponds to a multiscale representation based on powers of two. Accordingly, the minimum Δk to cover the frequency domain without holes is

$$\Delta k \geq \frac{2}{3} k_0 \tag{13}$$

This corresponds to a minimal choice of $\beta = 1$ octave.

5 Orientation coverage

To generate a filter with an orientation θ (measured from the positive horizontal axis), we can rotate the vertically oriented filter (Eq. 1) by $\theta - 90°$ with respect to the filter center (positive angle means counterclockwise rotation):

$$f(x, y, \psi, \theta) = A \cdot \frac{1}{2\pi\sigma_x\sigma_y} \exp\left(-\frac{x_\theta^2}{2\sigma_x^2} - \frac{y_\theta^2}{2\sigma_y^2}\right) \operatorname{cis}(k_0 x_\theta + \psi) \tag{14}$$

$$\begin{cases} x_\theta = x\cos(\theta - 90°) + y\sin(\theta - 90°) \\ y_\theta = -x\sin(\theta - 90°) + y\cos(\theta - 90°) \end{cases}$$

Note: k_0 can be considered the *radial peak frequency* (and the corresponding frequencies projected in the Cartesian space are $k_{0x} = k_0\cos(\theta - 90°)$ and $k_{0y} = k_0\sin(\theta - 90°)$).

6 Orientation bandwidth

@ cut-off frequencies (k_y^l, k_y^h) corresponding to half of the amplitude spectrum $F(k_x, k_y, \psi)$:

$$\Delta\theta = 2\arctan\frac{\sqrt{2\ln 2}}{k_0\sigma_y} \tag{15}$$

@ cut-off frequencies (k_x^l, k_x^h) at one standard deviation of the amplitude spectrum ($\sigma_f = 1/\sigma_x$):

$$\Delta\theta = 2\arctan\frac{1}{k_0\sigma_y} \tag{16}$$

By confounding the arc with the chord, we can derive the approximate relationships:

$$\Delta\theta \simeq \tfrac{2\sqrt{2\ln 2}}{\sigma_y} \qquad\qquad \Delta\theta \simeq \tfrac{2}{\sigma_y} \quad . \tag{17}$$

The minimum number (N) of orientations necessary to cover the angular frequency space is:

$$N \leq \frac{2\pi k_0}{\Delta\theta} \tag{18}$$

7 Spatial truncation

Although Gabor functions are well localized, they inevitably have infinite support. Truncation is used in most practical implementation to avoid aliasing. From practical considerations the spatial support can be chosen as a multiple ($2 \div 3$) of σ_x.

8 DC cancelation

In general the Gabor function does not integrate to zero and the DC component can be directly obtained from the Gabor spectrum (e.g., $A = 1$):

$$DC = F(0) = A \cdot \exp\left(-\frac{\sigma_x^2 k_0}{2}\right) \cos(\psi) \tag{19}$$

Note: only the sine component does have a null DC component. We further note that for a small bandwidth ($\beta < 0.7$) DC correction is not needed, as its value is close to the floating point precisiono (10^{-6}).

Basically, two approaches to remove the DC component can be followed: (1) modify the input signal in order to obtain a zero-mean visual signal, or (2) modify the filter's profile to obtain a zero-mean filter[2]. It is worthy to note that the subtraction of the DC value to the filter is not a proper solution.

9 Summary

Let us assume

$$\beta = 1 \;\rightarrow \Delta k = \frac{2}{3} k_0$$

we can find the following upper bound for k_0:

Nyquist limit:

$$k_0 + \frac{1}{2} \cdot \frac{2}{3} k_0 \leq \pi \;\rightarrow k_{0max} = \frac{3}{4}\pi$$

suggested value for k_0: $\frac{2}{3} k_{0max}$.

By adopting the cut-off frequency @ half of the amplitude spectrum, the corresponding lower bound for σ_x is

$$\sigma_{xmin} = \frac{4}{3\pi}\left(\frac{2+1}{2-1}\right)\sqrt{2\ln 2} = \frac{4}{\pi}\sqrt{2\ln 2}$$

Examples

- Gabor function parameters:

  $\beta = 1$, $A = 1$, $\sigma_y = \sigma_x$ (even though also $\sigma_y = 2\sigma_x$ will be explored)

[2] For phase-based feature extraction, the damaging effects the DC component are clear for the "direct methods" (by introducing a loss of balance between the convolutions with the even and odd Gabor filters), but they could not be a real problem for the "population methods".

$k_0 = \pi/2$, $\sigma_x = 6/\pi$ or $\sigma_x = 6\sqrt{2\ln 2}/\pi$

Spatial support: 11×11 pixels [-5:1:5] (or 21×21 when $\sigma_y = 2\sigma_x$)

- A set of 8 quadrature pairs of spatial filters are proposed:

$$h_k(x,y) = g(x,y,0,\theta_k) + jg(x,y,\pi/2,\theta_k)$$

with 8 filter orientations evenly distributed in the entire 180 degree range:

$$\theta_j \in \{0°, 22.5°, 45°, 67.5°, 90°, 112.5°, 135°, 157.5°\}$$

Suggested parameters On the basis of the daisy diagrams shown in Fig. 2, representing the Gabor band-pass channels, to have a good coverage of the 2-D spatial frequency domain, we suggest to fix k_0 and σ_x with respect to a cut-off frequency @ half of the amplitude spectrum, i.e.:

$$k_0 = \pi/2 \qquad\qquad \sigma_x = 6\sqrt{2\ln 2}/\pi$$

References

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[4] D. J. Fleet and A. D. Jepson. Computation of component image velocity from local phase information. *International Journal of Computer Vision*, 1:77–104, 1990.

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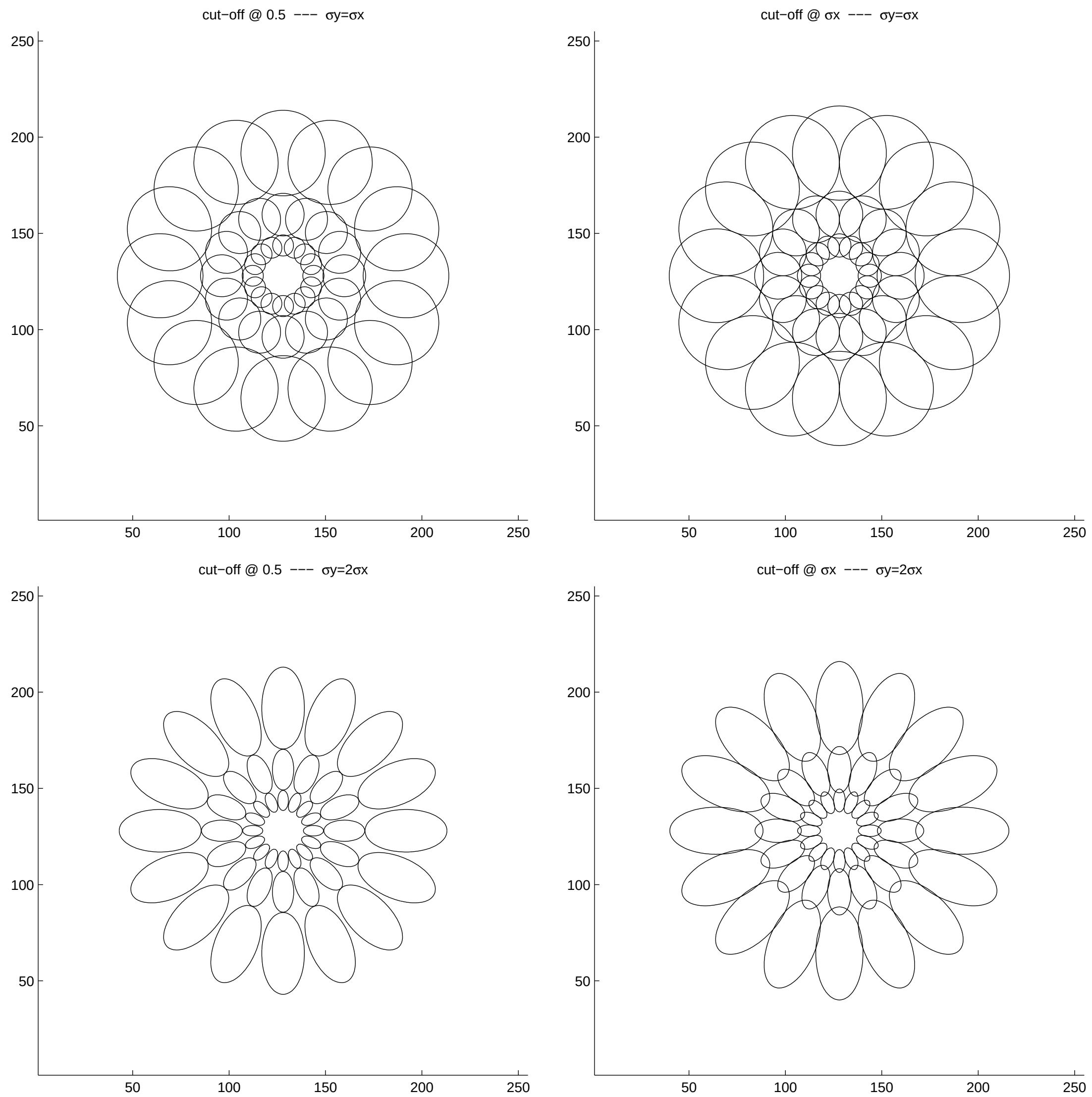

Figure 2: Band-pass frequency channels at 3 different scales for different choices of the cut-off frequency definition and different σ_y/σ_x ratio.